Non-Integrability and Chaos in Classical Cosmology

Amina Helmi (1) (2)* and Héctor Vucetich (2) †

(1) *Sterrewacht Leiden, University of Leiden, Niels Bohrweg 2, 2333 CA Leiden, The Netherlands.*

(2) *Facultad de Ciencias Astronómicas y Geofísicas, Universidad Nacional de La Plata, Paseo del Bosque s/n, 1900 La Plata, Argentina.*

A brief analysis of the dynamics of a Friedmann-Robertson-Walker universe with a conformally coupled, real, self-interacting, massive scalar field, based on the Painlevé theory of differential equations, is presented. Our results complete earlier works done within the framework of Dynamical System Theory.

We conclude that, in general, the system will not be integrable and that the chaos that has been found in a previous work, arises from the presence of movable logarithmic branch points in the solution in the complex plane of time.

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Several works in the latter years have tried to prove the importance of non-linearities in General Relativity [1–4] that may lead to the onset of chaos in the early stages of the universe. The chaotic behavior is even observed to occur in quite simplistic models like vacuum Bianchi IX (see [5–7]), due to only the coupling of the different scale factors, as this model does not contain any matter fields. The focus has also been put on the inflationary phase of the early universe, and on the role that different fields might have played in this scenario [8,9,3]. In many of these papers, the problem was attacked by means of numerical tools, based on Dynamical System Theory, which as we will show later in this brief report, do not explore some of the very interesting features of the system under study. Moreover, some people have shown (see [1] for a discussion of the problem) that the detection of chaotic behavior is quite dependent on the choice of the coordinate system, and thus methods which provide gauge independent (or covariant) measures or results about the behavior of the system under study are very much needed while working within theories of Gravitation. Lately [10] studies based on fractal basin boundaries have shown to be very useful, as no smooth coordinate transformation can undo a fractal pattern. On the other hand, from the analytical point of view, singularity studies can provide with very meaningful results, which

*ahelmi@strw.LeidenUniv.nl

†Member of the Carrera del Investigador de CONICET.

are independent of any particular gauge choice. And even more, they can be shown to be invariant under the group of homographic transformations of the dependent variables [11].

In this report we study the dynamics of a conformally coupled field in a homogeneous and isotropic universe. We shall consider a massive, self-interacting, scalar field conformally coupled to a Friedmann-Robertson-Walker metric [12]. The system is described by the following two degrees of freedom Hamiltonian:

$$H = \frac{1}{2}[-(p_a^2 + ka^2) + (p_\phi^2 + k\phi^2) + m^2a^2\phi^2 + \frac{\lambda}{2}\phi^4 + \frac{\Lambda}{2}a^4] \tag{1}$$

where a is the scale factor, k is the curvature and $\Lambda = \frac{2\Lambda_0}{3}$ is related to the cosmological constant. We have obviously considered a potential $V(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{1}{4}\hat{\lambda}\Phi^4$, where $\Phi = \phi/(a\sqrt{v})$, $\lambda = \hat{\lambda}/v$ is the self-interacting coefficient (v is the conformal volume) and m is the mass of the field. m^2 can take both positive and negative values, what would take into account processes of spontaneous symmetry breaking resulting in phase transitions which may have occurred in the early stages of the universe [13–16].

The Hamiltonian presented above (eq. (1)) is well-known and has been thoroughly studied within lattice-dynamics, condensed matter theory, field theory, etc. (see references in [17]). It is interesting to note that if we make the change of variable $a = -ix$, the Hamiltonian would correspond to that of a set of two quartically coupled non-linear oscillators [17], the study of which can be done analytically, giving a more complete view of the behavior of the system.

The analytical method is based on the Painlevé analysis of the differential equations, which states that if a solution is free from movable critical points other than poles, then it can be expressed as a generic Laurent series around one of its singularities. The test known as ARS algorithm [18], goes through the analysis of the leading order terms, the next order terms (or resonances) and the number of arbitrary constants that can be introduced in the expansion so as to make it generic. (It is important to note that the test gives only necessary conditions for the integrability of the system.)

When one studies the dominant behaviors of the system derived from eq. (1)

$$\ddot{a} + ka - m^2a\phi^2 - \Lambda a^3 = 0, \tag{2}$$

$$\ddot{\phi} + k\phi + m^2a^2\phi + \lambda\phi^3 = 0, \tag{3}$$

it is immediately possible to see that the leading order terms are independent of the value of the curvature k (see TABLE I), which is a very interesting and important result, as we shall interpret below. The cases for which the system passes P-test are very few. They correspond to special values of the parameters such as

$$\begin{cases} \Lambda = \lambda \text{ and } m^2 = -3\Lambda, \\ \Lambda = \lambda \text{ and } m^2 = -\Lambda, \end{cases} \quad \forall k, \tag{4}$$

$$\begin{cases} \Lambda = 16\lambda \text{ and } m^2 = -6\lambda, \\ \Lambda = 8\lambda \text{ and } m^2 = -3\lambda, \end{cases} \quad \text{for } k = 0. \tag{5}$$

In reference [17] the integrals of motion are given for these cases.

For the rest of the cases, which are the majority , the system shows different types of movable critical points: algebraic branch points, with irrational or complex values for the exponents, or logarithmic branch points. (See TABLE II. Notice that, indeed, for a very few values of the parameters Branch 1 will have integer resonances. In general the resonances will be irrational or complex, thus leading to the nonintegrability of the system. The same would happen in Branch 2, if ($r^2 = 1 - 8m^2/\lambda < 0$), or if r^2 is not the square of an integer or rational number. The same of course applies to Branch 3.)

It is interesting to compare these results with those obtained by Blanco et al. [19,20] by means of numerical methods. They analyzed the same Hamiltonian, but restricted themselves to the cases for which $k \neq 0$ (recall that $k = 0$ is a very important case indeed) and to a particular choice for the values of the parameters ($2m^2 + \lambda + \Lambda = 0$). The first restriction comes from a stability analysis which shows that the only fixed point for $k = 0$ is an elliptic point. This lead them to conclude, erroneously as we shall show, that a chaotic behavior cannot be reached in this flat universe. The second condition is a consequence of the Hamiltonian constraint $H = 0$ in General Relativity. With our approach, this constraint is always satisfied for each of the leading orders considered, without having to impose any restrictions on the values of the parameters. It is very easy to show with the method that we have been using, that the system passes the first two steps of the algorithm (leading order corresponds to Branch 1, the resonances are -1 and 4 with double multiplicity), but that the intented Laurent expansion will not have a sufficient number of arbitrary constants unless one introduces logarithmic terms in it. This fact is seen in the following: $-a_0^2 = b_0^2 = \frac{4}{|\Lambda - \lambda|}$, $ia_2 = \frac{k(\Lambda^2+\lambda^2-6\Lambda\lambda)}{6(5\Lambda^2+\lambda^2-6\Lambda\lambda)}(-a_0)$, $b_2 = \frac{kb_0}{6}$, $a_1 = a_3 = b_1 = b_3 = 0$, a_4 is arbitrary, and $b_4 = f(a_4)$. Thus a general solution would be of the form: $x(t) = ia(t) = a_0\tau^{-1} + a_2\tau + (a_4 + c_4 \log \tau)\tau^4 + ...$, and a similar expansion for $\phi(t)$.

Moreover, the chaos that has been observed by Blanco et al., can immediately be related to the presence of these movable logarithmic branch points. It is well known [21,22], that systems possessing this type of critical points display very interesting features in the complex plane of time. The singularities tend to accumulate in the complex $t-$plane, to form characteristic patterns such as chimneys (or spirals with increasing and decreasing radii for the case of complex resonances). It is also possible to show that the distance between neighboring singularities steadily decreases. Because of this, it is impossible to extend the solution further than a certain distance from a pair of initial singularities. Any path of analytic continuation appears to be trapped in a geometrically converging web of singularities that creates a natural boundary of the solution [23,24].

In synthesis, the above study has shown that there exists a relation between the occurrence of movable branch points (i.e. multivaluedness, formation of self-similar patterns which eventually will be natural boundaries if they lie on the same Riemann sheet) and the onset of chaos in a system. We should emphasize again that this result is independent of the value of k, and includes the case of a flat universe. This shows that there is no reason to believe that chaos will not be reached in this case.

Our results, based on the Painlevé analysis, show that for most of the values the parameters can take, the system is not integrable because of the presence of movable branch points, either algebraic (irrational and complex values for the resonances) or logarithmic. Besides, our results are independent of the curvature, as might have been expected, since it is well-known that the role of the curvature term can be neglected in Friedmann's equation during the inflationary phase. Thus, chaos is likely to be developed for all initial conditions of an homogeneous and isotropic universe, independently of the value of the curvature, cosmological and coupling constants, except for a zero measure set of values. The cases for which a chaotic stage may have never been reached are only a very few [25], that include for example when the coupling constant λ is equal to the cosmological constant Λ. This fact would in some sense rule out the very well-known initial-value problem of the very early states of the universe, if the time scale for which the chaotic behavior is completed is comparable to the characteristic time for which the inflationary phase is expected to last. If during the final stages of Λ-dominated inflation the scalar field were to oscillate around the potential minimum in a chaotic way, the particles produced in this process could be responsible for the subsequent reheating of the universe [26]. This reheating in a chaotic fashion might be then, the necessary one to produce a sufficiently high temperature so as to restore the scenario for the standard baryogenesis to take place.

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TABLE I. Leading order terms: $x \approx a_0\tau^p$, $\phi \approx b_0\tau^q$ (with $\tau = t - t_0$)

	p	q	a_0^2	b_0^2
Branch 1	-1	-1	$2\frac{(m^2+\lambda)}{m^4-\lambda\Lambda}$	$2\frac{m^2+\Lambda}{m^4-\lambda\Lambda}$
Branch 2	$\frac{1\pm\sqrt{1-8m^2/\lambda}}{2}$	-1	arbitrary	$-2/\Lambda$
Branch 3	-1	$\frac{1\pm\sqrt{1-8m^2/\Lambda}}{2}$	$-2/\lambda$	arbitrary

Branch 2 only appears if ($m^2 > 0$ and $\lambda > 0$), or if ($0 < -m^2 < \lambda$). And Branch 3, if ($m^2 > 0$ and $\Lambda > 0$), or if ($0 < -m^2 < \Lambda$).

TABLE II. Resonances: $x(t) = a_0\tau^p + a_1\tau^{p+r}$, etc.

	r
Branch 1	$x = \frac{3}{2} \pm \frac{1}{2(m^4-\lambda\Lambda)}\sqrt{-7m^8 - 16(\lambda+\Lambda)m^6 - 18\lambda\Lambda m^4 + 16\lambda\Lambda(\lambda+\Lambda)m^2 + 25\lambda^2\Lambda^2},\ -1,\ 4.$
Branch 2	$\mp\sqrt{1-8m^2/\lambda},\ -1,\ 0,\ 4$
Branch 3	$\mp\sqrt{1-8m^2/\Lambda},\ -1,\ 0,\ 4$